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# The central two-point connection problem for the Heun class of ODEs

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**Abstract.** We present a numerical approach to the central two-point connection problem for the confluent cases of the Heun class of differential equations. The crucial step is an ansatz for the solutions of the equations in terms of a generalized power series (called Jaffé expansions). It is shown that the resulting difference equations for the coefficients of these series are of Poincaré–Perron type. A (formal) asymptotic investigation of the solutions of these difference equations yields the exact eigenvalue condition.

## 1. Introduction

Boundary value problems for linear ordinary second-order differential equations with *singularities* at the endpoints of the interval of consideration are important in many problems of mathematical physics. Examples appear in quantum theory: the two Coulomb centres problem, the Stark effect in hydrogen, the anharmonic oscillator; they also appear in the Kerr and Teucholsky models in gravity theory etc. Since solutions of these problems are in general not polynomials, as is the case in simpler models leading to differential equations of hypergeometric type, a robust numerical procedure is needed to convert the computation of eigensolutions and corresponding eigenvalues by analytical means to appropriate standard problems of linear algebra. If the above-mentioned singularities are regular this procedure is well known. Complications arise if one or both of these points are irregular. In particular this is the case when a high precision of the result is required as is the case when exponentially small effects are considered.

Here, we present an approach valid for the latter case at rather general suppositions restricting practical applications to those equations which arise under confluence processes of the singularities of Heun's equation the Fuchsian differential equation with four (regular) singularities.

Suppose that there is an interval on the real axis at both endpoints of which is located a singularity of the underlying differential equation. If we look for parameters of the equation for which it has a solution behaving in a specified asymptotic manner while approaching the two singularities from inside the relevant interval simultaneously this means to solve a central two-point connection problem (CTCP). In [6–8] one of the authors has outlined a procedure for solving CTCP of the confluent cases of Heun's differential equation that was

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put into the context of boundary-eigenvalue problems of ordinary second-order differential equations [9].

The Heun class of differential equation arises from Heun's equation by means of confluence processes as already mentioned above (see [13, 12]) and comprises the Heun equation and four confluent cases. Recently, it has been studied intensively in [13].

The proposed procedure to deal with the CTCP consists of three steps.

(i) In a first step the equation is transformed by means of a gauge transformation of the dependent variable and a Möbius transformation of the independent variable to an appropriate form (Jaffé form).

(ii) In a second step the solution of the transformed equation is expanded in power series (Jaffé expansions); the coefficients of the Jaffé expansions are solutions of difference equations of Poincaré–Perron type.

(iii) In a third step further study of the asymptotic behaviour of the coefficients is undertaken resulting in the exact conditions for the eigenvalues.

Moreover, it is shown how to convert the problem into finite systems of linear equations by truncating the infinite procedure so that well known numerical methods can be applied in order to solve the connection problem. The error occurring by this truncation process decreases exponentially with increasing numbers of linear equations taken into account. This may be seen by considering the Birkhoff sets of the difference equations. The numerical practicability of the method has been checked by applying it to various physical systems as for example the quartic oscillator or a non-reflecting quantum potential (see e.g. [11, 2, 3, 10]).

In section 2 we briefly expose the formulation of the CTCP for the confluent cases of Heun's differential equation. Section 3 outlines the treatment of the difference equations on the basis of its Birkhoff sets. In section 4 we give the main result of this paper in the form of the eigenvalue condition for the connection problems in each of the confluent cases of Heun's equation.

## 2. The connection problems

### *2.1. The differential equations*

The confluent cases of Heun's differential equation with the appropriate choice of scaling of the independent variable and appropriate positioning of their singularities are as follows.

The single confluent case

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + \left[-p + \frac{c}{z} + \frac{d}{z+1}\right]\frac{\mathrm{d}y}{\mathrm{d}z} + \left[\frac{apz+\lambda}{z(z+1)}\right] y = 0 \tag{1}$$

the biconfluent case

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + \left[-p(z+1) + \frac{c}{z}\right]\frac{\mathrm{d}y}{\mathrm{d}z} + \left[-pa + \frac{\lambda}{z}\right] y = 0 \tag{2}$$

the triconfluent case

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + \left[-p(z^2-1)\right]\frac{\mathrm{d}y}{\mathrm{d}z} + \left[-paz+\lambda\right] y = 0 \tag{3}$$

and the double confluent case:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + \left[\frac{c}{z} - \frac{p(z^2-1)}{z^2}\right]\frac{\mathrm{d}y}{\mathrm{d}z} + \left[\frac{-paz+\lambda}{z^2}\right] y = 0. \tag{4}$$

In all of the above-written equations it is supposed that all the parameters are real and moreover the parameters $p$ and $c$ satisfy

$$p > 0 \qquad c > 1.$$

Confusions should not occur from using the same notations for the parameters in different equations here and below.

For each of the above-mentioned equations the solution of the CTCP is stated as follows. Consider the positive real axis. While approaching the points $z = 0$ and $z = \infty$ from inside the interval $[0, \infty[$ there are two different asymptotic behaviours (called dominant and recessive) of the solutions of each of equations (1)–(4). How to find the parameters $\lambda$ for which these differential equations have solutions that behave like the recessive solution while *simultaneously* approaching both points $z = 0$ and $z = \infty$ along the real axis. In fact, these values are the eigenvalues of the corresponding eigenvalue problem.

In the triconfluent case the point $z = 0$ is an ordinary point of the differential equation. It would be more natural to extend the interval on which the connection problem is treated from $[0, +\infty[$ to $]-\infty, +\infty[$. This can be done by applying the procedure given below twice: once for the interval $[0, \infty[$ and once for the interval $]-\infty, 0]$. In the latter case it is needed to replace $z$ by $-z$ and $p$ by $-p$. If we treat the connection problem on the whole real axis we speak of the *natural CTCP for the triconfluent case* of Heun's differential equation.

### *2.2. Linear transformations*

First we carry out the linear transformations of the dependent variable of each of the equations (1)–(4) according to

$$y = (z+1)^{-a} w. \tag{5}$$

In all cases we obtain from the original equation in the form (cf (1)–(4))

$$\frac{d^2 y(z)}{dz^2} + P(z)\frac{dy(z)}{dz} + Q(z)y(z) = 0 \tag{6}$$

the following differential equation

$$\frac{d^2 w(z)}{dz^2} + \left(P(z) - \frac{2a}{z+1}\right)\frac{dw(z)}{dz} + \left(Q(z) - \frac{aP(z)}{z+1} + \frac{a(a+1)}{(z+1)^2}\right)w(z) = 0. \tag{7}$$

The crucial difference between these two equations is that the multiplier in front of $w(z)$ in (7) can be estimated as

$$\tilde{Q}(z) = Q(z) - \frac{aP(z)}{z+1} + \frac{a(a+1)}{(z+1)^2} = \mathrm{O}(z^{-2}) \qquad \text{as } z \to \infty.$$

As a result of this fact first the point $z = 1$ becomes a reducible singularity (see [12]) in the case of equations (2)–(4) and, secondly, there exists a solution of equation (7) which has a finite limit when approaching the point at infinity along the positive real axis.

In the following we apply a Möbius transformation to the dependent variable of each confluent case of equation (7) according to

$$x = \frac{z - z_1}{z+1} \tag{8}$$

where the value of $z_1$ is zero in the single, bi-, and triconfluent case and is unity in the double confluent case.

In the following we give the results of the Möbius transformation for each confluent case.

Single confluent case

$$x(1-x)^2\frac{\mathrm{d}^2w}{\mathrm{d}x^2}[-px+(c+\{d-2(a+1)\}x)(1-x)]\frac{\mathrm{d}w}{\mathrm{d}x} \\ +[\lambda-ac+\{a(a+1)-ad\}x]w=0 \tag{9}$$

biconfluent case

$$x(1-x)^3\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+[-px+(1-x)^2\{c-2(a+1)x\}]\frac{\mathrm{d}w}{\mathrm{d}x} \\ +[\lambda+a(a+1)(1-x)\{x-ca\}]w=0 \tag{10}$$

triconfluent case

$$(1-x)^4\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+[-2px+p-2(a+1)(1-x)^3]\frac{\mathrm{d}w}{\mathrm{d}x} \\ +[\lambda-ap+a(a+1)(1-x)^2]w=0 \tag{11}$$

and double confluent case

$$(1-x^2)^2\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+[-8px+2(1-x^2)\{c-(a+1)(1+x)\}]\frac{\mathrm{d}w}{\mathrm{d}x} \\ +[4\lambda+\{a(a+1)(1+x)-2ac\}(1+x)-4pa]w=0. \tag{12}$$

Transformations (5), (8) taken together we call *Jaffé transformations*. It was Jaffé [5] who recognized its significance for the CTCP while calculating the spectrum of the ionized hydrogen molecule. Jaffé transformations can be easily extended to equations with more singularities as in our case, here. The resulting equations (9)–(12) are equations in *Jaffé (Jaffé–Lay) form*. This form is characterized by the following conditions.

(a) At zero there is either an ordinary point or a regular singularity of the equation.

(b) At infinity there is always a regular singularity of the equation (possibly reducible).

(c) Not more than two singularities lie within the unit circle $|x| \leqslant 1$.

(d) If there is one irregular singularity under consideration it lies at $x = 1$; if there are two they lie at $x = 1$ and at $x = -1$.

(e) If there is a regular singularity under consideration it lies at $x = 0$.

Each of the confluent cases (9)–(12) admit an ansatz of the form

$$w(x)=\sum_{n=0}^{n=\infty} g_n x^n. \tag{13}$$

The coefficients $g_n$ obey (irregular) difference equations of Poincaré–Perron type. This results from conditions (a) and (d) of the Jaffé–Lay form. The question arises as to what the conditions are on the solutions of the difference equation in order for the function (13) to be the eigensolution. Rigorously, the behaviour of the coefficients $g_n$ can be reconstructed from the formula inverse to (13)

$$g_n=\frac{1}{2\pi\mathrm{i}}\int_\Omega x^{-n-1}w(x)\mathrm{d}x. \tag{14}$$

for an appropriate contour $\Omega$ (see [1]).

## 3. Difference equations and Birkhoff sets

In the following we list these difference equations for each confluent case.

Single confluent case:

$$
\begin{aligned}
&g_{-1} = 0\\
&g_0 \text{ arbitrary}\\
&cg_1 + (\lambda - ac)g_0 = 0\\
&\left(1+\frac{\alpha_1}{n}+\frac{\beta_1}{n^2}\right)g_{n+1} + \left(-2+\frac{\alpha_0}{n}+\frac{\beta_0}{n^2}\right)g_n + \left(1+\frac{\alpha_{-1}}{n}+\frac{\beta_{-1}}{n^2}\right)g_{n-1} = 0 \qquad n \geqslant 1
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&\alpha_1 := 1 + c \qquad \beta_1 := c\\
&\alpha_0 := -p + d - c - 2a \qquad \beta_0 := \lambda - ac\\
&\alpha_{-1} := 2a - d - 1 \qquad \beta_{-1} := a(a+1) - d(a-1) - 2a.
\end{aligned}
$$

Biconfluent case:

$$
\begin{aligned}
&g_{-2} = g_{-1} = 0\\
&g_0 \text{ arbitrary}\\
&cg_1 + (\lambda - ca)g_0 = 0\\
&2(1+c)g_2 + (\lambda - ca - p - 2(1+a+c))g_1 + (a(a+1)+ca)g_0 = 0\\
&\left(1+\frac{\alpha_1}{n}+\frac{\beta_1}{n^2}\right)g_{n+1} + \left(-3+\frac{\alpha_0}{n}+\frac{\beta_0}{n^2}\right)g_n + \left(3+\frac{\alpha_{-1}}{n}+\frac{\beta_{-1}}{n^2}\right)g_{n-1}\\
&\qquad + \left(-1+\frac{\alpha_{-2}}{n}+\frac{\beta_{-2}}{n^2}\right)g_{n-2} = 0 \qquad n \geqslant 2
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&\alpha_1 := 1 + c \qquad \beta_1 := c\\
&\alpha_0 := -p - 2c - 2a + 1 \qquad \beta_0 := \lambda - ac\\
&\alpha_{-1} := c + 4a - 5 \qquad \beta_{-1} := a(a+1) + c(a-1) - 4a + 2\\
&\alpha_{-2} := -2a + 3 \qquad \beta_{-2} := -a(a+1) + 4a - 2.
\end{aligned}
$$

Triconfluent case:

$$
\begin{aligned}
&g_{-1} = g_{-2} = 0\\
&g_0, g_1 \text{ arbitrary}\\
&2g_2 + (p - 2(a+1))g_1 + ((\lambda - pa) + a(a+1))g_0 = 0\\
&6g_3 + (-8 + 2p - 4(a+1))g_2 + (-2p + 6(a+1)\\
&\qquad + (\lambda - pa) + a(a+1))g_1 - 2a(a+1)g_0 = 0\\
&\left(1+\frac{\alpha_2}{n}+\frac{\beta_2}{n^2}\right)g_{n+2} + \left(-4+\frac{\alpha_1}{n}+\frac{\beta_1}{n^2}\right)g_{n+1} + \left(6+\frac{\alpha_0}{n}+\frac{\beta_0}{n^2}\right)g_n\\
&\qquad + \left(-4+\frac{\alpha_{-1}}{n}+\frac{\beta_{-1}}{n^2}\right)g_{n-1} + \left(1+\frac{\alpha_{-2}}{n}+\frac{\beta_{-2}}{n^2}\right)g_{n-2} = 0 \qquad n \geqslant 2
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&\alpha_2 := 3 \qquad \beta_2 := 2\\
&\alpha_1 := p - 4 - 2(a+1) \qquad \beta_1 := p - 2(a+1)\\
&\alpha_0 := -2p + 6a \qquad \beta_0 := \lambda - pa + a(a+1)
\end{aligned}
$$

$\alpha_{-1} := -6(a+1) + 12 \qquad \beta_{-1} := 2(a+1)(3-a) - 8$
$\alpha_{-2} := 2a - 3 \qquad \beta_{-2} := (a+1)(a-4) + 6.$

Double confluent case:

$$g_{-1} = g_{-2} = 0$$
$$g_0, g_1 \text{ arbitrary}$$
$$2g_2 + 2(c-a-1)g_1 + (4(\lambda - pa) + a(a+1) - 2ca)g_0 = 0$$
$$6g_3 + 4(c-a-1)g_2 + \{-8p - 2(a+1) + 4(\lambda - pa) + a(a+1) - 2ca\}g_1 + 2(a(a+1) - ca)g_0 = 0$$
$$\left(1 + \frac{\alpha_2}{n} + \frac{\beta_2}{n^2}\right) g_{n+2} + \left(\frac{\alpha_1}{n} + \frac{\beta_1}{n^2}\right) g_{n+1} + \left(-2 + \frac{\alpha_0}{n} + \frac{\beta_0}{n^2}\right) g_n + \left(\frac{\alpha_{-1}}{n} + \frac{\beta_{-1}}{n^2}\right) g_{n-1} + \left(1 + \frac{\alpha_{-2}}{n} + \frac{\beta_{-2}}{n^2}\right) g_{n-2} = 0 \qquad n \geqslant 2 \tag{18}$$

$\alpha_2 := 3 \qquad \beta_2 := 2$
$\alpha_1 := 2(c-a-1) \qquad \beta_1 := 2(c-a-1)$
$\alpha_0 := -8p - 2a \qquad \beta_0 := 4(\lambda - pa) + a(a+1) - 2ca$
$\alpha_{-1} := -2(c-a-1) \qquad \beta_{-1} := -2(a-1)(c-a-1)$
$\alpha_{-2} := 2a - 3 \qquad \beta_{-2} := (a-1)(a-2).$

For each of these equations there exist formal solutions the number of which coincides with the order of the equation. They represent linear independent particular solutions of the corresponding difference equation asymptotically for $n \to \infty$ [4] and are called Birkhoff solutions [18]. The totality of the Birkhoff solutions of a difference equation is called a Birkhoff set [18]. In the following we write the Birkhoff sets $s_m$ for each of equations (15)–(18).

Single confluent case:

$$s_m(n) = \exp\left(\gamma_m n^{\frac{1}{2}}\right) n^{r_m} \left[1 + \frac{C_{m1}}{n^{\frac{1}{2}}} + \frac{C_{m2}}{n^{\frac{2}{2}}} + \cdots\right] \qquad m = 1, 2. \tag{19}$$

Biconfluent case:

$$s_m(n) = \exp(\gamma_{m1} n^{\frac{2}{3}} + \gamma_{m2} n^{\frac{1}{3}}) n^{r_m} \left[1 + \frac{C_{m1}}{n^{\frac{1}{3}}} + \frac{C_{m2}}{n^{\frac{2}{3}}} + \cdots\right] \qquad m = 1, 2, 3. \tag{20}$$

Triconfluent case:

$$s_m(n) = \exp(\gamma_{m1} n^{\frac{3}{4}} + \gamma_{m2} n^{\frac{1}{2}} + \gamma_{m3} n^{\frac{1}{4}}) n^{r_m} \left[1 + \frac{C_{m1}}{n^{\frac{1}{4}}} + \frac{C_{m2}}{n^{\frac{2}{4}}} + \cdots\right] \qquad m = 1, 2, 3, 4. \tag{21}$$

Double confluent case:

$$s_m(n) = \varrho_m^n \exp(\gamma_m n^{\frac{1}{2}}) n^{r_m} \left[1 + \frac{C_{m1}}{n^{\frac{1}{2}}} + \frac{C_{m2}}{n^{\frac{2}{2}}} + \cdots\right] \qquad m = 1, 2, 3, 4. \tag{22}$$

The factors in front of the brackets on the right-hand sides of (19)–(22) are called the asymptotic factors of the Birkhoff sets. In the following we give the coefficients for the asymptotic factors of the Birkhoff sets (19)–(22).

Single confluent case:

$$\begin{aligned}\gamma_1 &= 2p^{1/2}\\ \gamma_2 &= -\gamma_1\\ r_1 &= r_2 = a - 1 - \frac{c+d}{2} =: r.\end{aligned} \tag{23}$$

Biconfluent case:

$$\begin{aligned}\gamma_{m1} &= \frac{3}{2}\exp\left(\frac{2\pi\mathrm{i}m}{3}\right)p^{1/3}\\ \gamma_{m2} &= -\frac{3}{2}\exp\left(\frac{4\pi\mathrm{i}m}{3}\right)p^{2/3}\\ r_1 &= r_2 = r_3 = \frac{c-2a+4}{3} =: r.\end{aligned} \tag{24}$$

Triconfluent case:

$$\begin{aligned}\gamma_{m1} &= \frac{4}{3}\exp\left(\frac{2\pi\mathrm{i}m}{4}\right)p^{1/4}\\ \gamma_{m2} &= -\frac{1}{2}\exp\left(\frac{4\pi\mathrm{i}m}{4}\right)p^{1/2}\\ \gamma_{m3} &= -\frac{19}{24}\exp\left(\frac{6\pi\mathrm{i}m}{4}\right)p^{3/4}\\ r_1 &= r_2 = r_3 = r_4 = \frac{a-3}{2} =: r.\end{aligned} \tag{25}$$

Double confluent case:

$$\begin{aligned}&\varrho_m = 1 \qquad m = 1, 2\\ &\varrho_m = -1 \qquad m = 3, 4\\ &\gamma_{m1} = \exp(\pi\mathrm{i}m)(8p)^{1/2} \qquad m = 1, 2, 3, 4\\ &r_1 = r_2 = -1 + a - \frac{c}{2}\\ &r_3 = r_4 = -2 - \frac{c}{2}.\end{aligned} \tag{26}$$

As one can see from (23)–(26) the asymptotic factors of the Birkhoff sets depend only on the parameters $p, a, c, d$ of the differential equations (1)–(4) and do not depend upon the accessory parameter $\lambda$.

The general solutions of the difference equations (15)–(18) can be presented asymptotically for $n \to \infty$ as

$$g_n \sim \sum_{i=1}^{j} L_i s_i(n). \tag{27}$$

The upper limit of the sum in (27) is $j = 2$ in the single confluent case, $j = 3$ in the biconfluent case, and $j = 4$ in the tri- and double confluent cases. The coefficients $L_i$ depend on the parameters of the differential equations (1)–(4) and on the initial data for the solutions but not on the index variable.

The crucial point now is the fact that we can distinguish two characteristic behaviours of the solutions of the difference equations (15)–(18) for $n \to \infty$. According to the Birkhoff sets (19)–(22), and (23)–(26) we obtain—under rather weak conditions—exclusively exponentially increasing and exponentially decreasing solutions. The increasing solutions we call *dominant* and the decreasing ones we call *recessive*. It is important for the following to distinguish the dominant solution growing the fastest. We call it *maximum solution*. The CTCP for the confluent cases of Heun's differential equation is solved if we can find the parameters of the differential equation for which the coefficient(s) $L_{i_{\max}}$ of the maximum solutions of the difference equations (15)–(18) in (27) vanish. This will be expanded in the next section.

In the following we suppose that the Birkhoff set (19) for the single confluent case consists of one exponentially increasing and one exponentially decreasing asymptotic solution. In the biconfluent case we suppose that the Birkhoff set (20) includes one exponentially increasing and two exponentially decreasing solutions. In the triconfluent case we suppose that the Birkhoff set (21) includes one exponentially decreasing and three exponentially increasing solutions. In the double confluent case we suppose that the Birkhoff set (22) includes two exponentially increasing and two exponentially decreasing solutions.

## 4. The eigenvalue conditions

Our procedure of dealing with the CTCP for the confluent cases of Heun's differential equation results in irregular difference equations of Poincaré–Perron type, the order of which is the sum of the $s$-ranks of the irregular singularities in the underlying differential equation.

As we have seen in the preceding section there are specific fundamental systems of the difference equations which are characterized by the index-asymptotic behaviour for $n \to \infty$. These specific systems of particular solutions may be used to formulate the eigenvalue condition.

### *4.1. The single, bi- and triconfluent cases*

As one can see from the coefficients of the leading terms in the Birkhoff sets, there is in the single, bi-, and triconfluent cases one particular solution of the difference equations that is the maximum solution (as defined above) and will be denoted by $s_1$ in (19)–(22). We shall call the other solutions recessive. The eigenvalue conditions can be stated as follows. In all the above-mentioned confluent cases of Heun's differential equation the CTCP is solved when the maximum solution of the resulting difference equation is vanishing:

$$L_1(\lambda; p, a, \ldots) = 0.$$

In the single confluent case one can prove this by infinite continued fraction methods. In the biconfluent case it may be seen from Weierstrass' convergence criterion and Abel's limiting value theorem. In the triconfluent case one has to apply asymptotic methods of analysis since the series (13) are no longer convergent at $x = 1$ even in the case when we have an eigensolution. This is carried out in more detail in [11].

### *4.2. The double confluent case*

The difference equation here is of fourth order. However, because of the specific character of the Jaffé transformation we have two maximum solutions denoted by $s_1$ and $s_3$ in (22) and (26). Thus, the eigenvalue condition here is given by

$$L_1(\lambda; p, a, c) = L_3(\lambda; p, a, c) = 0. \tag{28}$$

Obviously, in this case we need a further parameter in order to meet these two conditions. This is given by the initial parameter $g_1$ in (18) which serves as an eigenvalue parameter if we would like to keep $g_0$ arbitrary for normalizing reasons.

*4.2.1. A reduction process.* The double confluent case is the only one in which we obtained two eigenvalue conditions. It is possible to reduce these to only one on the numerical level as below.

We consider equation (18) in the form

$$\begin{aligned} &A_0^{(0)}g_0 + A_1^{(0)}g_1 + A_2^{(0)}g_2 = 0 \\ &A_{-1}^{(1)}g_0 + A_0^{(1)}g_1 + A_1^{(1)}g_2 + A_2^{(1)}g_3 = 0 \\ &A_{-2}^{(n)}g_{n-2} + A_{-1}^{(n)}g_{n-1} + A_0^{(n)}g_n + A_1^{(n)}g_{n+1} + A_2^{(n)}g_{n+2} = 0 \qquad n \geqslant 2. \end{aligned} \tag{29}$$

It can also be written in the form of a system of infinite many linear equations

$$\boldsymbol{A} \cdot \boldsymbol{g} = \boldsymbol{0}. \tag{30}$$

The exact eigenvalue condition for the characteristic values may be written in the form

$$\det \boldsymbol{A} = 0. \tag{31}$$

The system (30) is truncated at a sufficiently large number $N$ of $n$ so that we obtain a system of $N+1$ linear equations

$$\boldsymbol{A}^{(N)} \cdot \boldsymbol{g}^{(N)} = \boldsymbol{0} \tag{32}$$

with

$$\boldsymbol{A}^{(N)} = \begin{bmatrix} g_0^{(0)} & g_1^{(0)} & g_2^{(0)} & 0 & 0 & 0 & 0 & \dots & & \\ g_{-1}^{(1)} & g_0^{(1)} & g_1^{(1)} & g_2^{(1)} & 0 & 0 & 0 & \dots & & \\ g_{-2}^{(2)} & g_{-1}^{(2)} & g_0^{(2)} & g_1^{(2)} & g_2^{(2)} & 0 & 0 & \dots & & \\ 0 & g_{-2}^{(3)} & g_{-1}^{(3)} & g_0^{(3)} & g_1^{(3)} & g_2^{(3)} & 0 & \dots & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ & \dots & 0 & g_{-2}^{(N-2)} & g_{-1}^{(N-2)} & g_0^{(N-2)} & g_1^{(N-2)} & g_2^{(N-2)} & 0 & 0 \\ & \dots & 0 & 0 & g_{-2}^{(N-1)} & g_{-1}^{(N-1)} & g_0^{(N-1)} & g_1^{(N-1)} & g_2^{(N-1)} & 0 \\ & \dots & 0 & 0 & 0 & g_{-2}^{(N)} & g_{-1}^{(N)} & g_0^{(N)} & g_1^{(N)} & g_2^{(N)} \end{bmatrix} \tag{33}$$

$$\boldsymbol{g}^{(N)} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_{N-1} \\ g_N \\ g_{N+1} \\ g_{N+2} \end{bmatrix}. \tag{34}$$

Since we have two exponentially increasing Birkhoff solutions in (22) truncation means that we have to set $g_{N+1} = 0$ and $g_{N+2} = 0$ in the system (32)–(34). The number of exponentially decreasing Birkhoff solutions in (22) is also two. With the initial conditions

$$\begin{aligned} g_N^{(1)} &= 1 \qquad & g_{N-1}^{(1)} &= 0 \\ g_N^{(2)} &= 0 \qquad & g_{N-1}^{(2)} &= 1 \end{aligned}$$

we obtain two linearly independent recessive solutions

$$g_n^{(1)}, g_n^{(2)} \qquad n = N, N-1, \ldots 0, -1, -2$$

by means of numerically stable backward recursions. The general recessive solution of (32)–(34) is then given by

$$g_n = K_1 g_n^{(1)} + K_2 g_n^{(2)}$$

with two arbitrary constants $K_1$ and $K_2$. The condition (31) is transformed by means of the truncation and the backward recursion process into

$$\det \begin{pmatrix} g_{-1}^{(1)} & g_{-1}^{(2)} \\ g_{-2}^{(1)} & g_{-2}^{(2)} \end{pmatrix} = 0$$

so that we eventually have to solve a $2\times 2$ determinant in order to calculate the characteristic parameters of the CTCP of the double confluent case of Heun's differential equation.

The eigenvalue condition of the double confluent case consists of two conditions (28). This is a consequence of the fact that the two relevant singularities are located on the unit circle thus two conditions have to be met one at each of the two endpoints of the relevant interval at $x = \pm 1$. This situation is compensated for by the fact that the solutions are expanded about an ordinary point of the differential equation and not about a (regular) singularity as is done in the other confluent cases. Mathematically, this is expressed by the initial equations of the difference equation (29) where one can see that it is recursively solvable only when $g_0$ *and* $g_1$ are chosen, thus there are two initial values. Fixing $g_0$ by normalizing reasons we see that we have one further eigenvalue parameter entering the problem in addition to the parameters of the differential equation. It was the aim of our numerical procedure outlined above to reduce the number of eigenvalue parameters from two to one. This was achieved by the backward recursion procedure from which we get the ratio $g_1/g_0$—playing the role of this second eigenvalue parameter of the CTCP—as a result!

## 5. Applications in physical sciences

In the foregoing sections we have presented a new and exact method to solve linear ordinary second-order differential equations with polynomial coefficients. Being obvious that it has a large field of applications, it is clear that the usefulness of the method is strictly dependent on how far it will be applicable to concrete problems and on whether it will allow us to obtain new or more accurate results. Therefore, we have made several investigations in this respect before publishing the method itself. The most important ones are [6–8, 16]. They deal with the most difficult cases the tri- and the double confluent ones and show that it is not only possible to calculate eigenvalues and eigensolutions but also to find new qualitative phenomena.

An example we briefly present here is the effect of 'avoided crossings' in the eigenvalue curves of the quartic oscillator. This is an exponentially small effect and—as far as we know—has not been calculated before. It is a natural CTCP of the triconfluent case related physically to the double-well potential. Besides the eigenvalue parameter $E$ we have the parameter $a$ describing the asymmetry of the two wells. In figure 1 we show a section of the eigenvalue curves of the quartic oscillator. The figure presents the six lowest-lying levels $E_n$, $n = 0, 1, \ldots, 5$ in dependence on the asymmetry $a$ of the two wells. The effect of avoided crossings is clearly seen.

In figure 2 the eigensolutions for the three lowest-lying eigenvalues are shown. On the left the potential is symmetric ($a = 0$) and on the right the potential is non-symmetric

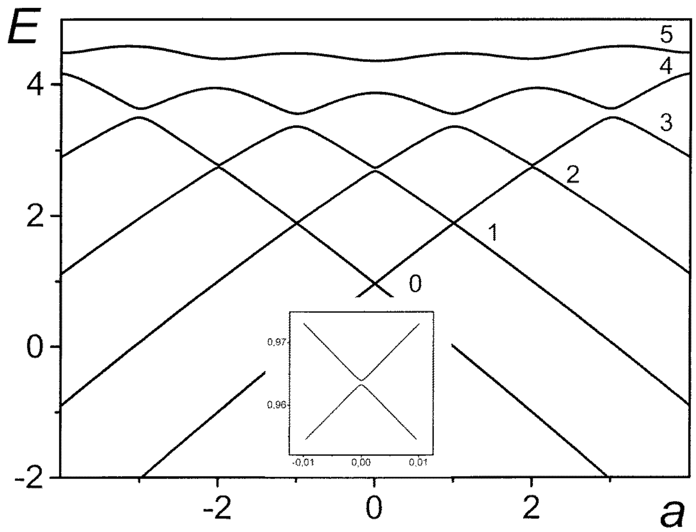


**Figure 1.** Effect of avoided crossings in the spectrum of the quantum quartic oscillator. Presented are the six lowest-lying energy levels in dependence on the asymmetry parameter $a$. The inset shows how the curve $E_0$ and $E_1$ come close to one another but do not cross.

$(a = 1)$. The full lines are the exact eigensolutions while the dotted ones are the results of a Ritz approximation on the basis of harmonic oscillators. For details the reader should consult the cited publications.

A second physical example is a non-reflecting potential that can be modelled by the double confluent case of Heun's equation. Here, in contrast to the quartic oscillator, the effect of avoided crossings is suppressed. There is one potential well on the negative real axis and the other on the positive one with a tunnelling barrier between them. A separate spectrum for each of the wells can be calculated. Besides the eigenvalue parameter $\lambda$ we once again have a parameter $a$ describing the asymmetry of the two wells. However, as we could show, there is an intrinsic symmetry of the differential equation so that after a rescaling $\lambda, a \to \tilde{\lambda}, a'$ the eigenvalue curves $\tilde{\lambda}(a')$ are symmetric with respect to $a' = 0$. Among the eigensolutions we found generalized polynomials which have not been discovered before, since they are beyond the set of classical orthogonal polynomials. Moreover, in view of asymptotics this example has led us to discover that although we have Stokes and anti-Stokes lines there are solutions which do not show the Stokes phenomenon [10].

A final statement concerns the role of Heun's differential equation and its confluent cases in the theory of Hamiltonian systems. As shown [17] there is a relation between the Heun class of equations (linear) and the class of the Painlevé equations (nonlinear): if a quantum system is described by any differential equation of Heun's class then the corresponding classical system is described by one of the differential equations of Painlevé's class. This sheds new light onto the role of Heun's equations and gives us the opportunity to use our calculations to qualitative characteristic features of Painlevé's transcendents.

## 6. Conclusions

The procedure given above is a general one in the sense that it is valid for ordinary second-order differential equations with polynomial coefficients. The characteristic feature of our

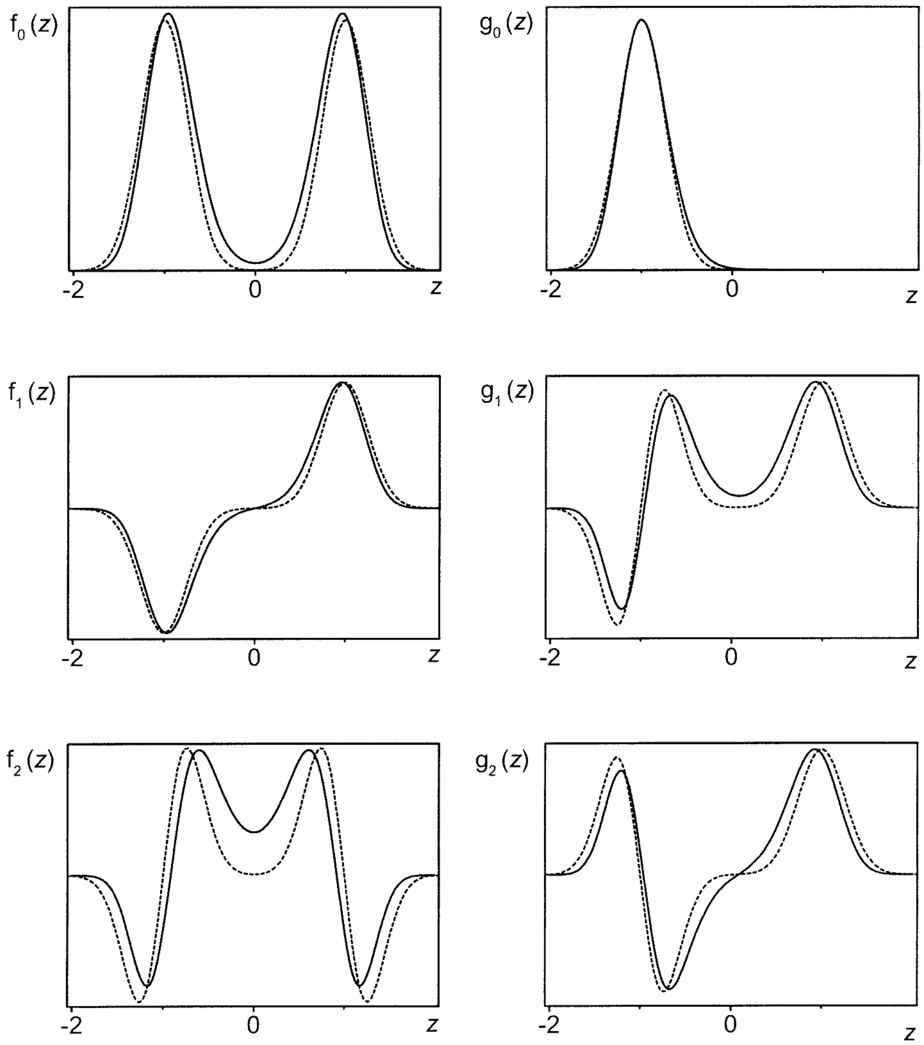


**Figure 2.** Exact (full curves) and approximated eigensolutions for the three lowest-lying energy levels of the quartic oscillator. On the left-hand side there is no asymmetry thus $a = 0$ while $a = 1$ on the right-hand side.

procedure is that it converts the differential equation into a difference equation of Poincaré–Perron type. The underlying idea is that the relevant interval is transformed by means of a Möbius transformation onto the real axis between zero and one (between $-1$ and $+1$ for the double confluent case) in such a manner that no other singularity of the differential equation than the relevant ones are within the unit circle. We are left with the above-mentioned difference equations which are characterized with respect to the connection problem by means of the asymptotic behaviour of their particular solutions. From these we obtain the exact eigenvalue conditions for all of the confluent cases of Heun's equation as the vanishing of the maximum solution (the two maximum solutions in the double confluent case). As a result, we can apply this method numerically by means of a truncation of the difference equation to an algebraic system of equations from which we know that the error is exponentially small with respect to the number of equations of the resulting matrix. Eventually, we should mention, that our method may be considered as an alternative to the so-called infinite-determinant method established by Schmidt *et al* [14, 15]. Unfortunately, there does not exist a numerical realization of the latter method, therefore quantitative results are not available and thus cannot be compared with our own results.

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